ST402 Principals and Methods of Statistical Practice
Mid-term Test

Answer as much of this as you can in the 45 minutes available. You may consult your notes but not your colleagues. All questions will be marked out of 10 (although question 4 may be slightly longer than the others).

1. I have an urn with 3 disks of which 2 are green and 1 is red. I toss a fair coin twice and count the number of heads; X is the number of heads. I then draw X disks from the urn without replacement; Y is the number of red disks that I draw.

   (a) Find $E(X)$ and $E(Y|X)$. Hence find $E(Y)$.

   (b) Find the probability mass function of Y.

2. Let X and Y be random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} kx, & 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

   (a) Find k.

   (b) Write down the density of $X|Y = y$ and hence find $\mathrm{Var}(X|Y = \frac{1}{2})$.

3. Let Y and Z be independent standard normal random variables.

   (a) Show that the moment generating function of Y is $M_Y(t) = e^{t^2/2}$ and hence write down the joint MGF $M_{Y,Z}(t,u)$.

   (b) Let $W = Y$ and $X = \rho Y + \sqrt{(1-\rho^2)}Z$. Derive the joint moment generating function of W and X. Show that X is a standard normal random variable.

4. Suppose that $X_j \sim iN(0, \sigma_X^2)$ and $Y_j \sim iN(\mu_Y, \sigma_Y^2)$ for $j = 1, \ldots, n$.

(a) Find the r^{th} moment of X.

(b) Find the expected value and variance of the estimator of σ_X^2

$$\hat{\sigma}_X^2 = \frac{1}{n} \sum_{j=1}^{n} X_j^2$$

(c) Find the expected value of the the estimator of σ_Y^2

$$\hat{\sigma}_Y^2 = \frac{1}{n} \sum_{j=1}^{n} (Y_j - \overline{Y})^2$$

where $\overline{Y} = \frac{1}{n} \sum_{j=1}^{m} Y_j$.